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Some Applications of Quarternions.

By

Toshizô Matsumoto.

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1. Riccati's Equation and Cayley's Formula.

The general solution of Riccati's equation

$$\frac{dz}{dt} = Lz^2 + Mz + N,$$

where L, M and N are functions of t alone, has the form

$$z = \frac{\alpha z_0 + \beta}{\gamma z_0 + \delta}, \qquad \cdots\cdots\cdots (1)$$

in which α, β, γ and δ are functions of t, and z_0 is an arbitrary constant. We can always assume $\alpha\delta - \beta\gamma = 1$. Conversely, given α, β, γ and δ, we can build an equation which z satisfies. For that purpose, solving the equation (1) with respect to z_0,

$$z_0 = \frac{\delta z - \beta}{-\gamma z + \alpha}.$$

Differentiating both sides with respect to t and noticing that

$$\frac{dz_0}{dt} = 0, \text{ and } \alpha\delta - \beta\gamma = 1,$$

we have the equation

$$\frac{dz}{dt} = (\gamma\delta' - \gamma'\delta)z^2 + \{(\delta\alpha' - \delta'\alpha) + (\beta\gamma' - \beta'\gamma)\}z + (\alpha\beta' - \alpha'\beta), \quad (2)$$

where the dashes mean the differentiation with respect to t. We now consider the form of the equation (2) for Cayley's formula,

$$z = \frac{(d+ic)z_0 - (b-ia)}{(b+ia)z_0 + (d-ic)}$$

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1 Klein: Ikosaeder p. 34. (e.g.)

where a, b, c, d are real constants satisfying the condition

$$a^2+b^2+c^2+d^2 = 1.$$

Consider these constants as real functions of the real variable t; in this case the equation (2) becomes

$$\frac{dz}{dt} = -irz + \frac{q-ip}{2} + \frac{q+ip}{2} z^2, \qquad (3)$$

where $\gamma\delta'-\gamma'\delta = (b+ia)(d'-ic')-(b'+ia')(d-ic)$

$$= bd'-b'd+ac'-a'c$$

$$+ i(ad'-a'd+cb'-c'b)$$

$$\equiv \frac{q+ip}{2},$$

$$\alpha\beta'-\alpha'\beta = \bar{\delta}(-\bar{\gamma}')-\bar{\delta}'(-\bar{\gamma})$$

$$= \overline{\gamma\delta'-\gamma'\delta}$$

$$= \frac{q-ip}{2},$$

lastly $\delta\alpha'-\delta'\alpha+\beta\gamma'-\beta'\gamma = 2i\{(dc'-d'c)+(ab'-a'b)\}$

$$\equiv -ir.$$

Or

$$\frac{p}{2} = ad'-a'd+b'c-bc'$$

$$\frac{q}{2} = ac'-a'c+bd'-b'd \qquad (4)$$

$$\frac{r}{2} = a'b-ab'+cd'-c'd$$

Hence *the function*

$$z = \frac{(d+ic)z_0-(b-ia)}{(b+ia)z_0+(d-ic)}$$

is the general solution af such an equation as

$$\frac{dz}{dt} = -irz + \frac{q-ip}{2} + \frac{q+ip}{2} z^2,$$

where $a^2+b^2+c^2+d^2 = 1.$

2. Darboux-Riccati's Fquation.

Consider two systems of rectangular axes, having the same origin; the one space-fixed and the other rotating. Let α, β, γ be the direction-cosines of any one of the space-fixed axes referred to the moving system; p, q, r, three components of rotation of the moving axes referred to themselves; then the kinematical equations are

$$\left.\begin{aligned} \frac{d\alpha}{dt} &= \beta r - \gamma q \\ \frac{d\beta}{dt} &= \gamma p - \alpha r \end{aligned}\right\} \qquad (1)$$

$$\frac{d\gamma}{dt} = \alpha q - \beta p,$$

where t is the time.

To discuss the solutions of the equations Prof. Darboux[1] proceeded as follows:—

We notice the condition

$$\alpha^2+\beta^2+\gamma^2 = \text{constant}.$$

When the constant is not equal to zero, we may always assume

$$\alpha^2+\beta^2+\gamma^2 = 1, \qquad (2)$$

which is consistent with our assumption that α, β, γ are the direction-cosines of an axis. Put

$$\left.\begin{aligned} \frac{\alpha+i\beta}{1-\gamma} &= \frac{1+\gamma}{\alpha-i\beta} \equiv x \\ \frac{\alpha-i\beta}{1-\gamma} &= \frac{1+\gamma}{\alpha+i\beta} \equiv -\frac{1}{y} \end{aligned}\right\} \qquad (3)$$

which give $\alpha = \dfrac{1-xy}{x-y}$, $\beta = i\dfrac{1+xy}{x-y}$, $\gamma = \dfrac{x+y}{x-y}$;

then the equatian (1) become

$$\frac{dx}{dt} = -irx + \frac{q-ip}{2} + \frac{q+ip}{2}x^2,$$

$$\frac{dy}{dt} = -iry + \frac{q-ip}{2} + \frac{q+ip}{2}y^2.$$

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1 Lecons sur la Théorie Générale des Surfaces, I (first edition) pp. 21–22.

Or writing σ for x or y,

$$\frac{d\sigma}{dt} = -ir\sigma + \frac{q-ip}{2} + \frac{q+ip}{2}\sigma^2, \tag{4}$$

which is the same equation as §1 (3). Hence the general solution is of the form

$$\sigma = \frac{(d+ic)\sigma_0 - (b-ia)}{(b+ia)\sigma_0 + (d-ic)},$$

and the relations between p, q, r and a, b, c, d are given by the equations §1 (4). Those equations may be transformed by some easy calculations into the following

$$\left.\begin{aligned} &\frac{da}{dt} + \frac{p}{2}d + \frac{q}{2}c - \frac{r}{2}b = 0, \\ &\frac{db}{dt} - \frac{p}{2}c + \frac{q}{2}d + \frac{r}{2}a = 0, \\ &\frac{dc}{dt} + \frac{p}{2}b - \frac{q}{2}a + \frac{r}{2}d = 0, \\ &\frac{dd}{dt} - \frac{p}{2}a - \frac{q}{2}b - \frac{r}{2}c = 0; \end{aligned}\right\} \tag{5}$$

noticing that $a^2+b^2+c^2+d^2 = 1$, therefore $a\frac{da}{dt} + b\frac{db}{dt} + c\frac{dc}{dt} + d\frac{dd}{dt} = 0$.

3. Given p, q, r to find a, b, c, d, we have to solve the system of four differential equations (5) of the last section. If we put $d=0$, then the first three equations become as follows:

$$\frac{da}{dt} = b\frac{r}{2} - c\frac{q}{2},$$

$$\frac{db}{dt} = c\frac{p}{2} - a\frac{r}{2},$$

$$\frac{dc}{dt} = a\frac{q}{2} - b\frac{p}{2},$$

which are of the same type as the system of equations (1) of the last section. Therefore the system (5) may be considered as a more general case and consequently we attempt, by some method, to obtain Riccati's equation which is equivalent to the system of differential equa-

tions (5). For that purpose we use Hamilton's quarternions $1, i_1, i_2, i_3$ which satisfy the relations

$$i_1^2 = i_2^2 = i_3^2 = i_1 i_2 i_3 = -1,$$

$$i_1 i_2 + i_2 i_1 = i_2 i_3 + i_3 i_2 = i_3 i_1 + i_1 i_3 = 0.$$

Now multiplying $1, i_1, i_2, i_3$ into the equations (5) and summing them up, we obtain after some easy calculations

$$\frac{dx}{dt} = x\left(\frac{p}{2} i_3 + \frac{q}{2} i_2 - \frac{r}{2} i_1\right), \tag{1}$$

where $$x = a + bi_1 + ci_2 + di_3.$$

This equation may be considered a special Riccati's equation. In a space of two dimensions, the analogous system of equations obtained from (1) § 2 by the assumption $\gamma = 0$, viz.,

$$\left.\begin{aligned} \frac{d\alpha}{dt} &= \beta r \\ \frac{d\beta}{dt} &= -\alpha r. \end{aligned}\right\} \tag{2}$$

Multiplying $1, i$ and summing them up, we have

$$\frac{dx}{dt} = -irx, \qquad x = \alpha + i\beta. \tag{3}$$

which is quite similar to (1).

4. As a more general case than § 2 (5), we consider the system of five equations

$$\left.\begin{aligned} \frac{d\alpha}{dt} &= \beta r - \gamma q - \delta p + \varepsilon q, \\ \frac{d\beta}{dt} &= \gamma p - \delta q + \varepsilon r - \alpha r, \\ \frac{d\gamma}{dt} &= -\delta r + \varepsilon r + \alpha q - \beta p, \\ \frac{d\delta}{dt} &= \varepsilon q + \alpha p + \beta q + \gamma r, \\ \frac{d\varepsilon}{dt} &= -\alpha q - \beta r - \gamma r - \delta q. \end{aligned}\right\} \tag{1}$$

From these equations we have

$$\alpha^2+\beta^2+\gamma^2+\delta^2+\varepsilon^2 = \text{const.},$$

and this constant may be regarded as unity except in the case of zero; i.e.

$$\alpha^2+\beta^2+\gamma^2+\delta^2+\varepsilon^2 = 1.$$

We put, after Prof. Darboux, § 2 (3),

$$\left.\begin{aligned} \frac{\alpha+\beta i_1+\gamma i_2+\delta i_3}{1-\varepsilon} &= \frac{1+\varepsilon}{\alpha-\beta i_1-\gamma i_2-\delta i_3} \equiv x, \\ \frac{\alpha-\beta i_1-\gamma i_2-\delta i_3}{1-\varepsilon} &= \frac{1+\varepsilon}{\alpha+\beta i_1+\gamma i_2+\delta i_3} \equiv -y^{-1}, \end{aligned}\right\} \quad (2)$$

whence we have

$$\alpha+\beta i_1+\gamma i_2+\delta i_3 = (1-\varepsilon)\,x,$$

$$\alpha-\beta i_1-\gamma i_2-\delta i_3 = (1+\varepsilon)\,x^{-1}.$$

Therefore

$$\frac{d\alpha}{dt}+\frac{d\beta}{dt}i_1+\frac{d\gamma}{dt}i_2+\frac{d\delta}{dt}i_3 = (1-\varepsilon)\frac{dx}{dt}-\frac{d\varepsilon}{dt}x,$$

which, by the equations (1), is equal to the following sum

$$\begin{aligned} &\beta r-\gamma q-\delta p+\varepsilon q+(\gamma p-\delta q+\varepsilon r-\alpha r)\,i_1 \\ &+(-\delta r+\varepsilon r+\alpha q-\beta p)\,i_2+(\varepsilon q+\alpha p+\beta q+\gamma r)\,i_3 \\ &= \varepsilon\,(q+ri_1+ri_2+qi_3)+(1-\varepsilon)\,x\,(-ri_1+qi_2+pi_3). \end{aligned}$$

In the same way,

$$\begin{aligned} \frac{d\alpha}{dt}-\frac{d\beta}{dt}i_1-\frac{d\gamma}{dt}i_2-\frac{d\delta}{dt}i_3 &= -(1+\varepsilon)\,x^{-1}\frac{dx}{dt}\,x^{-1}+\frac{d\varepsilon}{dt}\,x^{-1} \\ &= \varepsilon\,(q-ri_1-ri_2-qi_3)+(1+\varepsilon)(ri_1-qi_2-pi_3)\,x^{-1}. \end{aligned}$$

Hence we have

$$\begin{aligned} &\frac{d\alpha}{dt}+\frac{d\beta}{dt}i_1+\frac{d\gamma}{dt}i_2+\frac{d\delta}{dt}i_3+x\left(\frac{d\alpha}{dt}-\frac{d\beta}{dt}i_1-\frac{d\gamma}{dt}i_2-\frac{d\delta}{dt}i_3\right)x \\ &= -2\varepsilon\frac{dx}{dt}^{*} \\ &= 2\varepsilon x\,(ri_1-qi_2-pi_3)+\varepsilon\,(q+ri_1+ri_2+qi_3)+\varepsilon x\,(q-ri_1-ri_2-qi_3)\,x, \end{aligned}$$

* *The product of a scalar and a qnarternion is permutable.*

whence we have

$$\frac{dx}{dt} = -x(ri_1 - qi_2 - pi_3) - \frac{q + ri_1 + ri_2 + qi_3}{2} - x\frac{q - ri_1 - ri_2 - qi_3}{2}\, x. \quad (3)$$

Starting from the equations

$$\alpha + \beta i_1 + \gamma i_2 + \delta i_3 = -(1+\varepsilon)y,$$

$$\alpha - \beta i_1 - \gamma i_2 - \delta i_3 = -(1-\varepsilon)y^{-1},$$

we arrive at the same equation for y; hence x and y are the solutions of the following equation

$$\frac{d\sigma}{dt} = -\sigma(ri_1 - qi_2 - pi_3) - \frac{q + ri_1 + ri_2 + qi_3}{2} - \sigma\frac{q - ri_1 - ri_2 - qi_3}{2}\,\sigma. \quad (4)$$

This equation is quite analogical with § 2 (4)

$$\frac{d\sigma}{dt} = -ir\sigma + \frac{q - ip}{2} + \frac{q + ip}{2}\,\sigma^2.$$

5. Serret-Frenet's Formula.

Serret-Frenet's formulas of a space curve are discussed as a special case of the kinematical equations § 2 (1). Formulas for hyperspace[1] e.g., of five dimensions are

$$\left.\begin{aligned} \frac{d\alpha}{ds} &= \frac{\beta}{R_1}, \\ \frac{d\beta}{ds} &= \frac{\gamma}{R_2} - \frac{\alpha}{R_1}, \\ \frac{d\gamma}{ds} &= \frac{\delta}{R_3} - \frac{\beta}{R_2}, \\ \frac{d\delta}{ds} &= \frac{\varepsilon}{R_4} - \frac{\gamma}{R_3}, \\ \frac{d\varepsilon}{ds} &= -\frac{\delta}{R_4}, \end{aligned}\right\} \quad (1)$$

where s is the arc length, and $\frac{1}{R}$'s are curvatures. This system of equations does not admit of the application of quarternions. For that

1 G. E. A. Brunnel, Math. Ann., 19 (1882). F. Meyer, Jahresbericht d. Deut. Math. Ver., 19 (1910); &c.

purpose, we must lead these equations to those § 4 (1). But the equations (1) are a special case of a kinematical equations for hyperspace; and we rather consider the reduction of the latter equations.

The kinematical equations for n dimensions are given by the following system of n equations

$$\frac{da_i}{dt} = \sum_{h=1}^{n} p_{ih}\, a_h^* \qquad i = 1, 2, \ldots\ldots n. \tag{2}$$

where $$p_{ii} = 0, \qquad p_{ih} = -p_{hi},$$

so that the coefficients p make a skew symmetric determinant. Now we transform these equations by the following orthogonal transformations.

$$\left.\begin{aligned} \beta_j &= \sum_{k=1}^{n} a_{jk}\, a_k, \qquad |a_{jk}| = +1, \\ & \qquad\qquad j = 1, 2, \ldots n, \\ \text{or} \quad a_k &= \sum_{j=1}^{n} a_{jk}\, \beta_j, \qquad k = 1, 2, \ldots n. \end{aligned}\right\} \tag{3}$$

Differentiating with respect to t,

$$\frac{d\beta_j}{dt} = \sum_{k=1}^{n}\left(a_{jk}\frac{da_k}{dt} + a_k \frac{da_{jk}}{dt}\right),$$

by (2) $$= \sum_{k=1}^{n}\left(a_{jk}\sum_{h=1}^{n} p_{kh}\, a_h + a_k \frac{da_{jk}}{dt}\right)$$

by (3) $$= \sum_{k=1}^{n}\left(a_{jk}\sum_{h=1}^{n} p_{kh}\sum_{r=1}^{n} a_{rh}\,\beta_r + \sum_{r=1}^{n} a_{rk}\,\beta_r \frac{da_{jk}}{dt}\right)$$

$$= \sum_{r=1}^{n}\left(\sum_{h,k=1}^{n} p_{kh}\, a_{jk}\, a_{rh} + \sum_{k=1}^{n} a_{rk}\,\frac{da_{jk}}{dt}\right)\beta_r.$$

Put $$P_{jr}' \equiv \sum_{h,k=1}^{n} p_{kh}\, a_{jk}\, a_{rh},$$

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* T. Craig, *Displacements depending on One, Two and Three Parameters in a space of Four Dimensions*, Amer. J. Math., **20** (1898); for n dimensions, N. J. Hatzidakis, *Ibid.*, **22** (1900).

$$P_{jr}'' \equiv \sum_{k=1}^{n} a_{rk} \frac{da_{jk}}{dt},$$

and $$P_{jr} \equiv P_{jr}' + P_{jr}''.$$

Then we have

$$P_{jr}' = -\sum_{k,h=1}^{n} p_{hk}\, a_{rh}\, a_{jk} = -P_{rj}',$$

$$P_{jr}'' = -\sum_{k=1}^{n} a_{jk} \frac{da_{rk}}{dt} = -P_{rj}'';$$

hence $$P_{jr} = -P_{rj}, \text{ and } P_{jj} = 0.$$

Therefore the transformed equations are

$$\left.\begin{aligned} &\frac{d\beta_i}{dt} = \sum_{r=1}^{n} P_{jr}\, \beta_r, \\ &P_{jr} = -P_{rj}, \qquad j = 1, 2, \ldots n, \\ &P_{jj} = 0, \qquad\quad r = 1, 2, \ldots n. \end{aligned}\right\} \tag{4}$$

where

When the transformed equations are required to be

$$\left.\begin{aligned} &\frac{d\beta_j}{dt} = \sum_{r=1}^{n} q_{jr}\, \beta_r, \\ &q_{rj} = -q_{rj}, \qquad j = 1, 2, \ldots n, \\ &q_{jj} = 0, \qquad\quad r = 1, 2, \ldots n. \end{aligned}\right\} \tag{5}$$

Then considering $a's$ as unknown, we have to solve the system of n^2 differential equations

$$P_{jr} = q_{jr}, \qquad j = 1, 2, \ldots n,$$
$$r = 1, 2, \ldots n;$$

or more completely

$$\sum_{k=1}^{n} a_{rk} \frac{d\bar{a}_{jk}}{dt} + \sum_{h,k=1}^{n} p_{kh}\, a_{jk}\, a_{rh} = q_{jr}, \qquad j, r = 1, 2, \ldots n.$$

Or multiplying a_{rs} into both sides and summing with respect to r

$$\sum_{r=1}^{n}\sum_{k=1}^{n} a_{rs}\,a_{rk}\,\frac{da_{jk}}{dt}+\sum_{r=1}^{n}\sum_{h\,k=1}^{n} p_{kh}\,a_{rs}\,a_{jk}\,a_{rh}=\sum_{r=1}^{n} a_{rs}\,q_{jr},$$

or

$$\frac{da_{js}}{dt}+\sum_{k=1}^{n} p_{ks}\,a_{jk}=\sum_{r=1}^{n} a_{rs}\,q_{jr},$$

or

$$\frac{da_{js}}{dt}=\sum_{k=1}^{n}(p_{sk}\,a_{jk}+q_{jk}\,a_{ks}), \qquad j,s=1,2,\dots n. \tag{6}$$

On conditions that

$$\left.\begin{aligned} &\sum_{r=1}^{n} a_{jr}^{2}=1, && i=1,2,\dots n,\\ &\sum_{r=1}^{n} a_{ir}\,a_{jr}=0, && j=1,2,\dots n;\ i\neq j. \end{aligned}\right\} \tag{7}$$

By these conditions the number of the differential equations is reduced to $\frac{n(n-1)}{2}$. After Cayley these n^2 quantities $a's$ can be represented by $\frac{n(n-1)}{2}$ essential parameters; considering these parameters as new unknowns, the equations (6) will be transformed into a system of $\frac{n(n-1)}{2}$ differential equations with $\frac{n(n-1)}{2}$ unknowns. Any particular solution will suffice to obtain $a's$ and consequently the orthogonal transformations (3).

Specially for $n=5$, we have

$$q_{12}=q_{25}=q_{35}=q_{43}=r,$$

$$q_{15}=q_{31}=q_{42}=q_{45}=q,$$

$$q_{23}=q_{41}=p,$$

then the system of differential equations

$$\frac{da_i}{dt}=\sum_{h=1}^{5} p_{ih}\,a_h, \qquad i=1,2,\dots 5,$$

may be transformed into the system § 4 (1) and hence into § 4 (4). (For the case $n=4$, we may treat similarly.) Serret-Frenets' formulas (1) are a special type of kinematical equations for which

$$p_{12} = \frac{1}{R_1}, \qquad p_{13} = p_{14} = p_{15} = 0,$$

$$p_{23} = \frac{1}{R_2}, \qquad p_{24} = p_{25} = 0,$$

$$p_{34} = \frac{1}{R_3}, \qquad P_{35} = 0,$$

$$p_{45} = \frac{1}{R_4}.$$

Hence, by the above theory, this system of equations may be transformed into § 4 (4). Thus Prof. Darbaux's method[1] admits theoretically of a similar extension[2]

6. Line Element on Hypersphere.

If ds_n denote the line element on the hypersphere of n dimensions, then for the hypersphere

$$\alpha^2+\beta^2+\gamma^2+\delta^2+\varepsilon^2=1, \qquad (1)$$

$$ds_5^2 = d\alpha^2+d\beta^2+d\gamma^2+d\delta^2+d\varepsilon^2. \qquad (2)$$

On the other hand, from the equations § 4 (2)

$$\alpha^2+\beta^2+\gamma^2+\delta^2 = -(1-\varepsilon)^2xy^{-1} = 1-\varepsilon^2,$$

or dividing by $(1-\varepsilon)$, and rearranging we have

$$\left.\begin{aligned} \varepsilon &= (x+y)(x-y)^{-1} \\ \text{and} \qquad 1-\varepsilon &= -2y\,(x-y)^{-1} = -2\,(x-y)^{-1}y.^{*} \end{aligned}\right\} \qquad (2)$$

Now $$d\alpha+d\beta i_1+d\gamma i_2+d\delta i_3 = (1-\varepsilon)\,dx-xd\varepsilon$$

$$= -2\,dx\,(x-y)^{-1}y-xd\varepsilon,$$

also $$d\alpha-d\beta i_1-d\gamma i_2-d\delta i_3=(1-\varepsilon)\,y^{-1}\,dy\,y^{-1}+y^{-1}\,d\varepsilon;$$

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1 *Loc. cit.*

2 Mr. Eiesland treated the same problem of integration of kinematical equations, from another point of view, in Amer. J. Math., **20** (1898); **28** (1906).

* *When the product of two quarternions is equal to a scalar, then the factors are permutable.*

hence $$d\alpha^2+d\beta^2+d\gamma^2+d\delta^2 = 4\,dx\,(x-y)^{-1}\,dy\,(x-y)^{-1}-xy^{-1}\,d\varepsilon^2 + 2\,(xy^{-1}\,dy-dx)(x-y)^{-1}\,d\varepsilon.$$

But $$d\varepsilon\,(x-y) = (\varepsilon-1)\left(\frac{1+\varepsilon}{\varepsilon-1}\,dy-dx\right) = (\varepsilon-1)(xy^{-1}\,dy-dx),$$

hence $$xy^{-1}\,dy-dx = (x-y)\,\frac{d\varepsilon}{\varepsilon-1}.$$

Therefore $$d\alpha^2+d\beta^2+d\gamma^2+d\delta^2=4\,dx\,(x-y)^{-1}\,dy\,(x-y)^{-1} - \frac{1+\varepsilon}{\varepsilon-1}\,d\varepsilon^2+2\,\frac{d\varepsilon^2}{\varepsilon-1},$$

or $$ds_5^2=d\alpha^2+d\beta^2+d\gamma^2+d\delta^2+d\varepsilon^2=4\,dx\,(x-y)^{-1}\,dy\,(x-y)^{-1}, \qquad (4)$$

which is quite analogical with

$$ds_3^2 = d\alpha^2+d\beta^2+d\gamma^2 = \frac{4\,dx\,dy}{(x-y)^2}\,^1$$

7. **Parametric Representation.**

By aid of x and y, the five variables $\alpha, \beta, \gamma, \delta, \varepsilon$ can be represented as in the case of three variables. Now add the equations

$$\left.\begin{aligned} \alpha+\beta i_1+\gamma i_2+\delta i_3 &= (1-\varepsilon)\,x,\\ \alpha-\beta i_1-\gamma i_2-\delta i_3 &= -(1-\varepsilon)\,y^{-1}. \end{aligned}\right\} \qquad (1)$$

Then we have $$2\,\alpha = (x-y^{-1})(1-\varepsilon),$$

by § 6 (3) $$= -2\,(x-y^{-1})\,y\,(xy-)^{-1}.$$

Hence $$\alpha = (1-xy)(x-y)^{-1}. \qquad (2)$$

Next from (1)

$$\alpha i_1-\beta-\gamma i_3+\delta i_2=(1-\varepsilon)\,xi_1,$$

$$\alpha i_1+\beta-\gamma i_3+\delta i_2 = -(1-\varepsilon)\,i_1 y^{-1}.$$

Adding we have $$2\,\beta = -(1-\varepsilon)(xi_1+i_1\,y^{-1}),$$

$$= 2\,(xi_1+i_1\,y^{-1})\,y\,(x-y)^{-1}.$$

Hence $$\beta = i_1\,(1-i_1\,xi_1\,y)(x-y)^{-1}. \qquad (3)$$

γ and δ may be obtained in the same way; hence, adding the first formula § 6 (3), we have

1 Darboux: *Loc. cit.* p. 30.

$$\left.\begin{aligned} \alpha &= (1-xy)(x-y)^{-1}, \\ \beta &= i_1(1-i_1 x i_1 y)(x-y)^{-1}, \\ \gamma &= i_2(1-i_2 x i_2 y)(x-y)^{-1}, \\ \delta &= i_3(1-i_3 x i_3 y)(x-y)^{-1}, \\ \varepsilon &= (x+y)(x-y)^{-1}. \end{aligned}\right\} \qquad (4)$$

For the hypersphere of four dimensions, we have to put $\varepsilon=0$, and hence $x=-y$,

$$\left.\begin{aligned} \alpha &= \frac{1}{2}(x^{-1}+x), \\ \beta &= \frac{1}{2}(i_1 x^{-1}-x i_1), \\ \gamma &= \frac{1}{2}(i_2 x^{-1}-x i_2), \\ \delta &= \frac{1}{2}(i_3 x^{-1}-x i_3); \end{aligned}\right\} \qquad (5)$$

for the sphere we have to put $i_2=i_3=0$, $i_1 \equiv i$,

$$\left.\begin{aligned} \alpha &= (1-xy)(x-y)^{-1}, \\ \beta &= i(1+xy)(x-y)^{-1}, \\ \varepsilon &= (x+y)(x-y)^{-1}; \end{aligned}\right\} \qquad (6)$$

for the circle, $i_2 = i_3 = \varepsilon = 0$, $\quad i_1 \equiv i$ and $x = -y$,

$$\left.\begin{aligned} \alpha &= \frac{1}{2}(x^{-1}+x), \\ \beta &= \frac{i}{2}(x^{-1}-x). \end{aligned}\right\} \qquad (7)$$

8. **Transformation of Line-element.**

The square of the line element ds^2 does not change by the substitution

$$x = Ax_1 B + C, \qquad y = Ay_1 B + C, \qquad (1)$$

where A, B, C are constant quarternions. For

$$ds^2 = 4\,dx(x-y)^{-1}\,dy\,(x-y)^{-1}.$$

$$= x\,Adx_1 B\{A(x_1-y_1)B\}^{-1}\,Ady_1 B\{A(x_1-y_1)B\}^{-1},$$

$$= 4\,A dx_1 (x_1-y_1)^{-1} dy_1 (x_1-y_1)^{-1} A_1.$$

Since ds^2 is a scalar

$$ds^2 = 4\,dx_1 (x_1-y_1)^{-1} dy_1 (x_1-y_1)^{-1}.$$

Next ds^2 does not change by the substitutions

$$x = x_1^{-1}, \qquad y = y_1^{-1}. \tag{2}$$

For since $(x-y)^{-1} = x_1 (y_1-x_1)^{-1} y_1 = y_1 (y_1-x_1)^{-1} x_1$,

$$ds^2 = dx_1 (x_1-y_1)^{-1} dy_1 (x_1-y_1)^{-1}.$$

Lastly ds_5^2 does not change by the substitutions

$$\left.\begin{aligned} x &= (Ax_1+B)(Cx_1+D)^{-1}, \\ y &= (Ay_1+B)(Cy_1+D)^{-1} \end{aligned}\right\} \tag{3}$$

where A, B, C, D are constant quarternions.

Since $$x = \{AC^{-1}(Cx_1+D)+B-AC^{-1}D\}(Cx_1+D)^{-1}$$

$$= AC^{-1}+(B-AC^{-1}D)(Cx_1+D)^{-1}.$$

Putting $$x_2 \equiv Cx_1+D, \qquad x_3 \equiv x_2^{-1},$$

then $$x \equiv AC^{-1}+(B-AC^{-1}D)\,x_3,$$

in the same way

$$y_2 \equiv Cy_1+D, \qquad y_3 \equiv y_2^{-1},$$

then $$y = AC^{-1}+(B-AC^{-1}D)\,y_3.$$

Therefore by (1) and (2), we may prove the invariance of ds_5^2. We remark that since

$$ds_5^2 = 4\,dx\,(x-y)^{-1}\,dy\,(x-y)^{-1},$$

$$= 4\,dy\,(x-y)^{-1}\,dx\,(x-y)^{-1},$$

ds_5^2 is also invariant by the transformations

$$\left.\begin{aligned} x &= (Ay_1+B)(Cy_1+D)^{-1}, \\ y &= (Ax_1+B)(Cx_1+D)^{-1}. \end{aligned}\right\} \tag{4}$$

9. **Meaning of the Parameter x.**

When $(\omega, \xi, \eta, \zeta)$ denote the coordinates of the projection of a point on the hypersphere

$$\alpha^2+\beta^2+\gamma^2+\delta^2+\varepsilon^2 = 1,$$

from its pole on the equatorial hyperspace of four dimensions, then by analogy, the equations of projection are

$$\frac{\alpha}{\omega} = \frac{\beta}{\xi} = \frac{\gamma}{\eta} = \frac{\delta}{\zeta} = 1-\varepsilon,$$

whence by easy calculations

$$\alpha = \frac{2\omega}{\omega^2+\xi^2+\eta^2+\zeta^2+1},$$

$$\beta = \frac{2\xi}{\omega^2+\zeta^2+\eta^2+\zeta^2+1},$$

$$\gamma = \frac{2\eta}{\omega^2+\xi^2+\eta^2+\zeta^2+1},$$

$$\delta = \frac{2\zeta}{\omega^2+\xi^2+\eta^2+\zeta^2+1},$$

$$\varepsilon = \frac{\omega^2+\xi^2+\eta^2+\zeta^2-1}{\omega^1+\xi^2+\eta^2+\zeta^2+1}.$$

Now $$\alpha+\beta i_1+\gamma i_2+\delta i_3 = \frac{2(\omega+\xi i_1+\eta i_2+\zeta i_3)}{\omega^2+\xi^2+\eta^2+\zeta^2+1},$$

$$= (1-\varepsilon)\, x,$$

hence $$= \frac{2}{\omega^2+\xi^2+\eta^2+\zeta^2+1}\, x.$$

Therefore $$x = \omega+\xi i_1+\eta i_2+\zeta i_3.$$

For a sphere, x is the complex variable on the Gauss' plane.

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